

# LOCAL CRITERIA FOR THE EXISTENCE OF AN ACCELERATED FRAME OF REFERENCE

S. Manoff

*Bulgarian Academy of Sciences  
Institute for Nuclear Research and Nuclear Energy  
Department of Theoretical Physics  
Blvd. Tzarigradsko Chaussee 72  
1784 Sofia - Bulgaria*

*E-mail address: smanov@inrne.bas.bg*

## Abstract

A local criteria for the existence of an accelerated frame of reference is found. An accelerated frame of reference could exist in all regions where a non-null (non-isotropic) vector field does not degenerate in a null (isotropic) vector field.

## 1 Introduction

One of the important problem arising in considerations of a frame of reference in spaces with affine connections and metrics is finding conditions for the existence of a frame of reference and especially for the existence of an accelerated frame of reference [2].

Every frame of reference [1] is described by a set of the following geometrical objects:

- (a) A differentiable manifold  $M$  ( $\dim M = n$ ) considered as a model of space or space-time.
- (b) A non-null (non-isotropic) contravariant vector field  $u$  [ $u \in T(M)$ ,  $g(u, u) = e \neq 0$ ].
- (c) A tangent subspace (hypersurface)  $T_x^{\perp u}(M)$ , orthogonal to  $u$  at every point  $x \in M$ , where  $u$  is defined. This hypersurface (subspace) is determined by the vector field  $u$  and its corresponding projective metrics  $h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u)$  and  $h^u = \bar{g} - \frac{1}{e} \cdot u \otimes u$  as well as by the orthogonal to  $u$  contravariant vector fields  $\xi_{(a)\perp}$  [ $\xi_{(a)\perp} \in T(M)$ ,  $a = 1, \dots, n-1$ ,  $g(u, \xi_{(a)\perp}) = 0$ ].
- (d) (Contravariant) affine connection  $\nabla = \Gamma$ . The affine connection  $\Gamma$  is related to the covariant differential operator  $\nabla_u$  along  $u$ , determining the transport of tensor fields along the vector field  $u$ .

For the existence of an accelerated frame of reference, the hypersurfaces determined by a given vector field  $u$  at different points of  $M$  should not intersect each other. The vector field  $u$  is usually interpreted as the velocity vector field of observers which trajectories belong to a congruence of lines [set of non-intersecting lines (curves)] with (at least one) parameter, related to the proper

time of the observer. Then the trajectories are called world lines of the observers. Every hypersurface, orthogonal to  $u$ , under this interpretation has constant proper time. The hypersurfaces at different points, where  $u$  is defined, have different constant proper times.

For one observer [one trajectory (curve) with tangent vector  $u$  at every of its points], it could happen that the hypersurfaces, orthogonal to  $u$  at different points of the curve, intersect each other. If the trajectory of the observer is interpreted as its world line (the parameter of the curve is related to its proper time) then the intersection points will not have a unique proper time. One and the same intersection point will have different proper times from the point of view of the different hypersurfaces to which it belongs. This ambiguity of the definition of the proper time at the intersecting points could be considered and interpreted in two different ways:

(a) The intersection points of hypersurfaces with different (constant) proper times are boundary points of the region where an accelerated frame of reference could exist. The reason for such assumption is that for  $\dim M = 4$  a hypersurface, orthogonal to an observer's world line, appears as the 3-dimensional real world at a given proper time of the observer. An intersection point is an event belonging to the 3-dimensional world at different moments of the proper time of the observer. The event cannot be distinguished in the time from point of view of the observer. Its existence could not be uniquely described by the use of a frame of reference since it is not determined uniquely in the time scale of the frame of reference.

(b) The intersection points of two hypersurfaces (with different constant proper times of an observer to which world line they are orthogonal) are events which existence could not be described by a time interval of the proper time of the observer. The notion of time interval for intersection points is meaningless: two events with different proper times on the world line of the observer would both appear simultaneously with an event at an intersection point of the two hypersurfaces with the corresponding proper time. This fact requires a closer investigation related to the conditions for existence of intersection points.

## 2 Local conditions for the existence of an intersection point

1. Let us consider two-parametric curves from the congruence  $x^i(\tau, \lambda)$  with the tangent vectors  $u$  and  $\xi_\perp$ , where  $g(u, \xi_\perp) = 0$ . Let the parameters of the curves be  $\tau$  and  $\lambda$  correspondingly. Then the tangent vectors  $u$  and  $\xi_\perp$  could be written in a co-ordinate basis in the forms

$$u = \frac{d}{d\tau} = \frac{dx^i}{d\tau} \cdot \partial_i = u^i \cdot \partial_i \quad , \quad \xi_\perp = \frac{d}{d\lambda} = \frac{dx^j}{d\lambda} \cdot \partial_j = \xi_\perp^j \cdot \partial_j \quad .$$

If a frame of reference is well defined in  $M$  then there should be no intersection points of the subspaces (hypersurfaces), orthogonal to  $u$  at different points of the curve with tangent vector  $u$ .

Let us investigate the opposite case when two hypersurfaces, orthogonal to  $u$  at different points of the curve  $x^i(\tau, \lambda = \text{const.})$ , intersect each other and violate in this way the unique definition of frame of reference.

Let two two-parametric sets of curves  $x^i(\tau, \lambda)$  be given with the curves  $x^i(\tau = \tau_0 = \text{const.}, \lambda)$  and  $x^i(\tau, \lambda = \lambda_0 = \text{const.})$ . Let us assume that a curve  $x^i(\tau = \tau_0 = \text{const.}, \lambda)$  intersects a curve  $x^i(\tau = \tau_0 + d\tau, \lambda)$  at a point  $B$  with co-ordinates  $\bar{x}^i = x^i(\tau, \lambda)$ . From a point  $A$  with co-ordinates  $x^i(\tau_0, \lambda_0)$  at the curve  $x^i(\tau = \tau_0 = \text{const.}, \lambda)$  the point  $B$ , lying along the same curve, let have the co-ordinates  $\bar{x}^i = x^i(\tau_0, \lambda_0 + d\lambda)$ . From a point  $C$  with co-ordinates  $x^i(\tau_0 + d\tau, \lambda_0)$  at the curve  $x^i = x^i(\tau_0 + d\tau, \lambda)$ , the point  $B$ , lying as intersecting point also at this curve, let have the co-ordinates  $\bar{x}^i = x^i(\tau + d\tau, \lambda_0 + k \cdot d\lambda)$ .

If the co-ordinates  $\bar{x}^i$  from both the points  $A$  and  $C$  of the curve  $x^i(\tau, \lambda_0)$  are identical (indistinguishable) then

$$\bar{x}^i = x^i(\tau_0, \lambda_0 + d\lambda) = x^i(\tau + d\tau, \lambda_0 + k \cdot d\lambda) \quad .$$

The co-ordinates of the point  $B$  could be expressed by the co-ordinates of the point  $A$  or of point  $C$  by the use of the exponent of the ordinary differential operator [3]. On the one side,  $\bar{x}^i$  could be expressed by the co-ordinates of point  $C$  and then the co-ordinates of point  $C$  could be expressed by the co-ordinates of the point  $A$ .

$$\begin{aligned} \bar{x}_{ACB}^i &= x^i(\tau + d\tau, \lambda_0 + k \cdot d\lambda) = \\ &= \{(\exp[k \cdot d\lambda \cdot \frac{d}{d\lambda}])x^i\}_{(\tau_0 + d\tau, \lambda_0)} = \\ &\quad \left\{(\exp[k \cdot d\lambda \cdot \frac{d}{d\lambda}]) \circ (\exp[d\tau \cdot \frac{d}{d\tau}])x^i\right\}_{(\tau_0, \lambda_0)} \quad . \end{aligned}$$

On the other side,  $\bar{x}^i$  could be expressed directly by the co-ordinates of the point  $A$

$$\begin{aligned} \bar{x}_{AB}^i &= x^i(\tau_0, \lambda_0 + d\lambda) = \\ &= \{(\exp[d\lambda \cdot \frac{d}{d\lambda}])x^i\}_{(\tau_0, \lambda_0)} \quad . \end{aligned}$$

Since  $\bar{x}_{ACB}^i$  and  $\bar{x}_{AB}^i$  should be identical with the co-ordinates of the point  $B$  from point of view of observers at point  $A$  and  $C$ , we have the condition

$$\bar{x}_{ACB}^i = \bar{x}_{AB}^i \quad .$$

Then

$$\left\{(\exp[k \cdot d\lambda \cdot \frac{d}{d\lambda}]) \circ (\exp[d\tau \cdot \frac{d}{d\tau}])x^i\right\}_{(\tau_0, \lambda_0)} = \{(\exp[d\lambda \cdot \frac{d}{d\lambda}])x^i\}_{(\tau_0, \lambda_0)} \quad .$$

The last expression could be written more explicitly up to the second order of  $d\tau$  and  $d\lambda$  as

$$\begin{aligned} &\{[1 + k \cdot d\lambda \cdot \frac{d}{d\lambda} + \frac{1}{2!} \cdot k^2 \cdot d\lambda^2 \cdot \frac{d^2}{d\lambda^2} + \dots] \circ \\ &\circ [1 + d\tau \cdot \frac{d}{d\tau} + \frac{1}{2!} \cdot d\tau^2 \cdot \frac{d^2}{d\tau^2} + \dots]x^i\}_{(\tau_0, \lambda_0)} = \end{aligned}$$

$$\begin{aligned}
&= \{[1 + k \cdot d\lambda \cdot \frac{d}{d\lambda} + \frac{1}{2} \cdot k^2 \cdot d\lambda^2 \cdot \frac{d^2}{d\lambda^2} + \\
&+ d\tau \cdot \frac{d}{d\tau} + k \cdot d\lambda \cdot d\tau \cdot \frac{d}{d\lambda} \circ \frac{d}{d\tau} + \frac{1}{2} \cdot d\tau^2 \cdot \frac{d^2}{d\tau^2} + \dots]x^i\}_{(\tau_0, \lambda_0)} \approx \\
&\approx \{[1 + d\lambda \cdot \frac{d}{d\lambda} + \frac{1}{2} \cdot d\lambda^2 \cdot \frac{d^2}{d\lambda^2}]x^i\}_{(\tau_0, \lambda_0)} \quad , \\
&\{[(k-1) \cdot d\lambda \cdot \frac{d}{d\lambda} + d\tau \cdot \frac{d}{d\tau} + \frac{1}{2} \cdot (k^2 - 1) \cdot d\lambda^2 \cdot \frac{d^2}{d\lambda^2} + \\
&+ k \cdot d\lambda \cdot d\tau \cdot \frac{d}{d\lambda} \circ \frac{d}{d\tau} + \frac{1}{2} \cdot d\tau^2 \cdot \frac{d^2}{d\tau^2}]x^i\}_{(\tau_0, \lambda_0)} \approx 0 \quad .
\end{aligned}$$

Up to the first order of  $d\tau$  and  $d\lambda$ , under the conditions  $d\lambda < \varepsilon_1 \ll 1$ ,  $d\tau < \varepsilon_2 \ll 1$ ,  $d\tau \cdot d\lambda < \varepsilon_1 \cdot \varepsilon_2 < \varepsilon^2 \ll 1$ , we obtain the relation

$$[(k-1) \cdot d\lambda \cdot \frac{dx^i}{d\lambda} + d\tau \cdot \frac{dx^i}{d\tau}]_{(\tau_0, \lambda_0)} \approx 0 \quad .$$

If we further assume that  $d\lambda = \bar{\varepsilon}_1 \ll 1$ ,  $d\tau = \bar{\varepsilon}_2 \ll 1$ , then

$$\begin{aligned}
[(k-1) \cdot \bar{\varepsilon}_1 \cdot \frac{dx^i}{d\lambda} + \bar{\varepsilon}_2 \cdot \frac{dx^i}{d\tau}]_{(\tau_0, \lambda_0)} &\approx 0 \quad , \\
[(k-1) \cdot \bar{\varepsilon}_1 \cdot \xi_{\perp}^i + \bar{\varepsilon}_2 \cdot u^i]_{(\tau_0, \lambda_0)} &\approx 0 \quad ,
\end{aligned}$$

$$u_{(\tau_0, \lambda_0)}^i = -\alpha \cdot \xi_{\perp(\tau_0, \lambda_0)}^i \quad , \quad \alpha = (k-1) \cdot \frac{\bar{\varepsilon}_1}{\bar{\varepsilon}_2} \quad .$$

The last expressions lead to the relation between the vectors  $u$  and  $\xi$  at the point  $A$ :

$$u_{(\tau_0, \lambda_0)} = -\alpha \cdot \xi_{\perp(\tau_0, \lambda_0)} \quad , \quad \alpha = (k-1) \cdot \frac{\bar{\varepsilon}_1}{\bar{\varepsilon}_2} \quad .$$

*Remark.* The factor  $k$  could be omitted if we chose  $(k-1) \cdot \bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \varepsilon$ . If  $\alpha = 0$  then  $k = 1$  and  $u_{(\tau_0, \lambda_0)} = 0$ . The last expression is in contradiction with the prerequisite for the existence of the two parametric congruence  $x^i = x^i(\tau, \lambda)$   $[(d\tau/d\lambda) = 0]$  at all considered points of  $M$ . If  $u = 0$  then  $dx^i/d\tau = 0$  and  $x^i = x^i(\lambda)$  at the considered point  $A$  with co-ordinates  $x^i(\tau_0, \lambda_0)$ .

If the contravariant vector field  $u$  is interpreted as the velocity of an observer then in the vicinity of the intersection point  $B$  of the hypersurface  $\tau = \tau_0 = \text{const.}$  with the hypersurface  $\tau = \tau_0 + \bar{\varepsilon}_2 = \text{const.}$  the velocity  $u$  of the observer will be proportional to the vector field  $\xi_{\perp}$ , tangential to the curve  $x^i = x^i(\tau_0 = \text{const.}, \lambda)$  and lying in the hypersurface  $x^i(\tau_0 = \text{const.}, \lambda)$ . Since the vector field  $\xi_{\perp}$  is orthogonal to  $u$ , i.e. since  $g(u, \xi_{\perp}) = 0$ , then from the condition  $u_{(\tau_0, \lambda_0)} = -\alpha \cdot \xi_{\perp(\tau_0, \lambda_0)}$ , it follows that

$$\begin{aligned}
g(u, \xi_{\perp})_{(\tau_0, \lambda_0)} &= -\alpha \cdot g(\xi_{\perp}, \xi_{\perp})_{(\tau_0, \lambda_0)} = 0 \quad , \quad \alpha \neq 0 \quad , \\
g(u, \xi_{\perp})_{(\tau_0, \lambda_0)} &= -\frac{1}{\alpha} \cdot g(u, u) = 0 \quad .
\end{aligned}$$

Therefore, the contravariant non-null (non-isotropic) vector fields  $u$  and  $\xi_{\perp}$   $[g(u, u) = e \neq 0, g(\xi_{\perp}, \xi_{\perp}) \neq 0, g(u, \xi_{\perp}) = 0]$  degenerate to null (isotropic)

vector fields  $[g(u, u) = e = 0, g(\xi_\perp, \xi_\perp) = 0, g(u, \xi_\perp) = 0]$  in the vicinity of the intersection point  $B$ .

If a pair  $(u, \xi_\perp)$  with  $g(u, \xi_\perp) = 0$  of two non-null vector fields  $u$  and  $\xi_\perp$  degenerate to a pair of null vector fields in the vicinity of a point belonging to a hypersurface (orthogonal to  $u$ ) then this point could be consider as an intersection point of two hypersurfaces with different constant proper times. The point  $A$  could be consider as a point where the conditions for a frame of reference are violated and from this point on, in the direction to the intersection point  $B$ , there exists no frame of reference. The point  $A$  could be considered as a boundary point at the hypersurface  $x^i(\tau_0 = const., \lambda)$  for which a frame of reference is not more defined. This fact could be interpreted physically as the existence of a limited velocity  $u$  such that  $g(u, u) = e = 0$  for which an observer could not have a unique determined proper frame of reference. On the one side, a frame of reference is defined for a non-null vector field  $u$  [ $g(u, u) = e \neq 0$ ] determining its corresponding projective metrics  $h_u = g - (1/e) \cdot g(u) \otimes g(u)$  and  $h^u = \bar{g} - (1/e) \cdot u \otimes u$  only if  $e \neq 0$ . On the other side, the existence of a frame of reference forbids the existence of intersection points of hypersurfaces, orthogonal to the velocity vector field of the observer. The last condition leads to the same requirement valid for the existence of projective metrics for the metric field  $u$ . Therefore, the degeneration of a non-null velocity vector field to a null velocity vector field is an obstacle for the existence of a local frame of reference for the regions and points where the null vector field is defined.

A hypersurface, orthogonal to the vector  $u$ , is determined by  $n - 1$  vectors  $\xi_{(a)\perp}$ ,  $a = 1, \dots, n - 1$ , orthogonal to  $u$ . Every of these vectors could lie at a curve having intersection point with a curve on an other hypersurface, orthogonal to  $u$  and with different proper time. All intersection points with their neighborhoods determine the boundaries of the region where a frame of reference could exist. Therefore, *a local criteria for the existence of a frame of reference is the existence of a region in space or space-time where the velocity vector field of an observer does not degenerate to a null-vector field.*

### 3 Conclusion

In the present paper local conditions are considered for the regions where a frame of reference could exist. It turned out that there is a simple condition which could be used as a criteria for the existence of a frame of reference: A frame of reference could exist for an observer until its velocity appears as a non-null (non-isotropic) vector field. In general relativity, this means that the velocity of an observer should be smaller than the velocity of light in vacuum if the observer wish to see and describe events around him in the frame of its own frame of reference. If the velocity of the observer reaches to the velocity of light then he would be unable to detect and describe the events around him. To our great surprise, this limitation appears in full correspondence with the conditions imposed in the definition of a frame of reference on the velocity vector field to which the corresponding projective metrics belong. It is worth to be mentioned that the local criteria for the existence of a frame of reference is valid in all spaces with affine connections and metrics which could be used as models of space or space-time.

## References

- [1] Manoff. S., *Frames of reference in spaces with affine connections and metrics*.  
Class. Quantum Grav. **18** (2001) 6, 1111-1125. E-print (1999) gr-qc/99 08 061
- [2] Hehl F. W., Lemke J., Mielke E. W., *Two Lectures on Fermions and Gravity*.  
Proc. of the School on Geometry and Theoretical Physics, Bad Honnef, 12-16.02. 1990. eds. Debrus J. and Hirshfeld A. C. (Springer Verlag, Berlin 1991)
- [3] Schutz B. F., *Geometrical methods of mathematical physics* (Cambridge University Press, Cambridge, 1982). Russian translation: (Mir, Moscow, 1984)